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Angular Momentum Gymnastics

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Books:

A.R. Edmonds: *Angular Momentum in Quantum Mechanics*
Princeton Paperbacks (20 – 25 €)

B.R. Judd: *Angular Momentum Theory for Diatiomic Molecules*
Academic Press (55 €)

M. Mizushima: *Theory of Rotating Diatomic Molecules*
John Wiley & Sons (58 – 453 €)

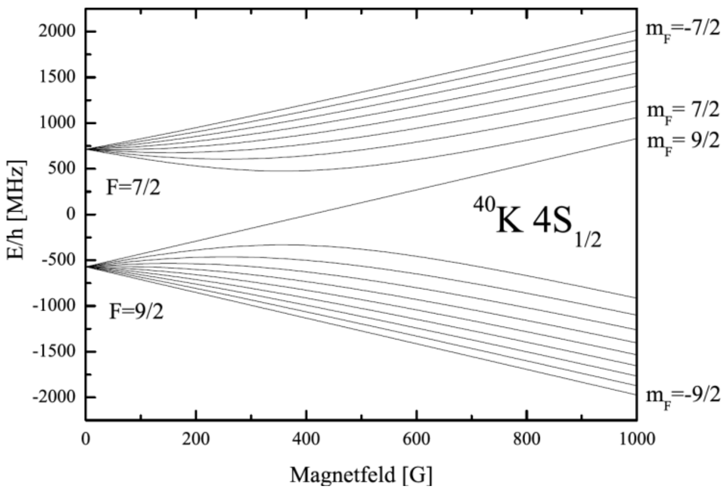
R.N. Zare: *Angular Momentum: Understanding Spatial Aspects in Chemistry and Physics*
Baker Lecture Series (110 €) **(Careful: incompatible phase convention)**

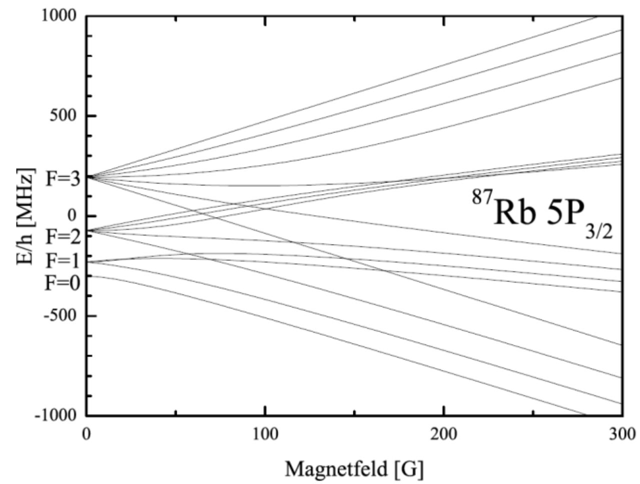
Metcalf, van der Straaten: *Laser cooling and Trapping*

(Strupid) Examples:

① Atomic decay channels / Rabi frequency

② Zeeman effect (and a little bit of Paschen-Back)





Diss. Klempt

③ Molecules: Rabi frequency / Stark effect / Hönl-London factor

Reduced matrix element, Wigner-Eckart

THE SPHERICAL TENSOR NOTATION. We define the spherical components[3] of a vector $\mathbf{r}$ as

$$r_{\pm 1} = \mp \frac{1}{\sqrt{2}}(x \pm iy); \qquad r_0 = z \tag{5.1.3}$$

i.e.

$$x = \frac{1}{\sqrt{2}}(r_{-1} - r_{+1}); \qquad y = \frac{i}{\sqrt{2}}(r_{-1} + r_{+1})$$

5.4. The Reduced Matrix Elements of a Tensor Operator

DEFINITION. It is convenient to define scalar quantities which differ slightly from the left-hand factor in (5.3.1). The *V-C* coefficient is replaced by one more symmetrical in the quantum numbers of initial and final states, by use of the symmetry relation (3.7.4). We have then the definition of the *reduced* or *double-bar* matrix elements

$$\begin{aligned}(\gamma'\ j'\ m'|T(k\ q)|\gamma\ j\ m) &= (-1)^{j-m}\frac{(j'\ m'\ j\ -m|j'\ j\ k\ q)}{(2k+1)^{\frac{1}{2}}}(\gamma'\ j'||\mathbf{T}(k)||\gamma\ j)\\ &= (-1)^{j'-m'}\begin{pmatrix} j' & k & j\\ -m' & q & m\end{pmatrix}(\gamma'\ j'||\mathbf{T}(k)||\gamma\ j)\\ &= (-1)^{k-j+j'}\frac{(k\ q\ j\ m|k\ j\ j'\ m')}{(2j'+1)^{\frac{1}{2}}}(\gamma'\ j'||\mathbf{T}(k)||\gamma\ j).\end{aligned} \tag{5.4.1}$$

The convention we have adopted is that of Racah (1942); it is compared with the conventions and notations used by some other workers in Table 5.1.

Reduced matrix element, Wigner-Eckart

Table 2.

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{\frac{1}{2}J} \left[\frac{(j_1 + j_2 - j_3)!(j_1 + j_3 - j_2)!(j_2 + j_3 - j_1)!}{(j_1 + j_2 + j_3 + 1)!} \right]^{\frac{1}{2}} \frac{(\frac{1}{2}J)!}{(\frac{1}{2}J - j_1)!(\frac{1}{2}J - j_2)!(\frac{1}{2}J - j_3)!}$$

if J is even.

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = 0 \quad \text{if} \quad J \text{ is odd} \quad \text{where} \quad J = j_1 + j_2 + j_3$$

$$\begin{pmatrix} J+\frac{1}{2} & J & \frac{1}{2} \\ M & -M-\frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad (-1)^{J-M-\frac{1}{2}} \left[\frac{J - M + \frac{1}{2}}{(2J + 2)(2J + 1)} \right]^{\frac{1}{2}} \qquad (J+\tfrac{1}{2},\, J,\, \tfrac{1}{2})$$

$$\begin{pmatrix} J+1 & J & 1 \\ M & -M-1 & 1 \end{pmatrix} \qquad (-1)^{J-M-1} \left[\frac{(J - M)(J - M + 1)}{(2J + 3)(2J + 2)(2J + 1)} \right]^{\frac{1}{2}}$$

$$\begin{pmatrix} J+1 & J & 1 \\ M & -M & 0 \end{pmatrix} \qquad (-1)^{J-M-1} \left[\frac{(J + M + 1)(J - M + 1) \cdot 2}{(2J + 3)(2J + 2)(2J + 1)} \right]^{\frac{1}{2}} \qquad (J+1,\, J,\, 1)$$

$$\begin{pmatrix} J & J & 1 \\ M & -M-1 & 1 \end{pmatrix} \qquad (-1)^{J-M} \left[\frac{(J - M)(J + M + 1) \cdot 2}{(2J + 2)(2J + 1)(2J)} \right]^{\frac{1}{2}}$$

$$\begin{pmatrix} J & J & 1 \\ M & -M & 0 \end{pmatrix} \qquad (-1)^{J-M} \frac{M}{[(2J + 1)(J + 1)J]^{\frac{1}{2}}} \qquad (J,\, J,\, 1)$$

Edmonds, the important part:

CHAPTER 7

The Evaluation of Matrix Elements in Actual Problems

7.1. Matrix Elements of the Tensor Product of Two Tensor Operators

$$\begin{aligned}(7.1.3)\quad &(\gamma' j_1' j_2' \| \mathsf{T}(k_1)\mathsf{U}(k_2) \| \gamma j_1 j_2)[(2k_1+1)(2k_2+1)]^{-\frac{1}{2}} \\ &= \sum_{\gamma''} (\gamma' j_1' j_2' \| \mathsf{T}(k_1) \| \gamma'' j_1 j_2') \\ &\qquad \times (\gamma'' j_1 j_2' \| \mathsf{U}(k_2) \| \gamma j_1 j_2)[(2k_1+1)(2k_2+1)]^{-\frac{1}{2}} \\ &= \sum_{m_1 m_1' m_2 m_2' q_1 q_2 Q} (j_1' m_1' j_1 m_1 | j_1' j_1 k_1 q_1)(j_2' m_2' j_2 m_2 | j_2' j_2 k_2 q_2) \\ &\qquad \times (k_1 q_1 k_2 q_2 | k_1 k_2 K Q)\cdot(-1)^{j_1+j_2+m_1+m_2} \\ &\qquad \times (\gamma' j_1' m_1' j_2' m_2' | X(K Q) | \gamma j_1 -m_1 j_2 -m_2)\end{aligned}$$

We see immediately that these two expressions are associated with different coupling schemes for the angular momenta $j_1' j_1, j_2', j_2$ and the left-hand sides are related by the corresponding transformation coefficient

$$\begin{aligned}(7.1.4)\quad &(\gamma' j_1' j_2' J' \| \mathsf{X}(K) \| \gamma j_1 j_2 J)(2K+1)^{-\frac{1}{2}} \\ &= \sum_{\gamma''} (\gamma' j_1' \| \mathsf{T}(k_1) \| \gamma'' j_1) \\ &\qquad \times (\gamma'' j_2' \| \mathsf{U}(k_2) \| \gamma j_2)[(2k_1+1)(2k_2+1)]^{-\frac{1}{2}} \\ &\qquad \times ((j_1' j_1)k_1, (j_2' j_2)k_2, K | (j_1' j_2')J', (j_1 j_2)J, K)\end{aligned}$$

Selection rules, branching ratios:

SINGLE OPERATOR IN COUPLED SCHEME. We obtain the reduced matrix element of a tensor operator $\mathbf{T}(k)$ working only on part 1 in the coupled scheme $(\gamma\, j_1\, j_2\, J\, M)$. We put $k_2 = 0$ in (7.1.5), substituting $\mathbf{U}(k) = 1$.

$$(\gamma'\, j_1'\, j_2\, J' || \mathbf{T}(k) || \gamma\, j_1\, j_2\, J) = (-1)^{j_1' + j_2 + J + k}[(2J+1)(2J'+1)]^{\frac{1}{2}} \begin{Bmatrix} j_1' & J' & j_2 \\ J & j_1 & k \end{Bmatrix} \times (\gamma'\, j_1' || \mathbf{T}(k) || \gamma\, j_1) \tag{7.1.7}$$

$$\begin{Bmatrix} a & b & c \\ 1 & c-1 & b-1 \end{Bmatrix} = (-1)^s \left[\frac{s(s+1)(s-2a-1)(s-2a)}{(2b-1)2b(2b+1)(2c-1)2c(2c+1)}\right]^{\frac{1}{2}}$$

$$\begin{Bmatrix} a & b & c \\ 1 & c-1 & b \end{Bmatrix} = (-1)^s \left[\frac{2(s+1)(s-2a)(s-2b)(s-2c+1)}{2b(2b+1)(2b+2)(2c-1)2c(2c+1)}\right]^{\frac{1}{2}}$$

$$\begin{Bmatrix} a & b & c \\ 1 & c-1 & b+1 \end{Bmatrix} = (-1)^s \left[\frac{(s-2b-1)(s-2b)(s-2c+1)(s-2c+2)}{(2b+1)(2b+2)(2b+3)(2c-1)2c(2c+1)}\right]^{\frac{1}{2}}$$

$$\begin{Bmatrix} a & b & c \\ 1 & c & b \end{Bmatrix} = (-1)^{s+1} \frac{2[b(b+1)+c(c+1)-a(a+1)]}{[2b(2b+1)(2b+2)2c(2c+1)(2c+2)]^{\frac{1}{2}}}$$

where $s = a + b + c$.

② Zeeman effect

Table 2.

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{\frac{1}{2}J} \left[\frac{(j_1 + j_2 - j_3)!(j_1 + j_3 - j_2)!(j_2 + j_3 - j_1)!}{(j_1 + j_2 + j_3 + 1)!} \right]^{\frac{1}{2}} \frac{(\frac{1}{2}J)!}{(\frac{1}{2}J - j_1)!(\frac{1}{2}J - j_2)!(\frac{1}{2}J - j_3)!}$$

if J is even.

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = 0 \quad \text{if} \quad J \text{ is odd} \quad \text{where} \quad J = j_1 + j_2 + j_3$$

$$\begin{pmatrix} J+\frac{1}{2} & J & \frac{1}{2} \\ M & -M-\frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad (-1)^{J-M-\frac{1}{2}} \left[\frac{J - M + \frac{1}{2}}{(2J + 2)(2J + 1)} \right]^{\frac{1}{2}} \qquad (J+\tfrac{1}{2}, J, \tfrac{1}{2})$$

$$\begin{pmatrix} J+1 & J & 1 \\ M & -M-1 & 1 \end{pmatrix} \qquad (-1)^{J-M-1} \left[\frac{(J - M)(J - M + 1)}{(2J + 3)(2J + 2)(2J + 1)} \right]^{\frac{1}{2}}$$

$$(J+1, J, 1)$$

$$\begin{pmatrix} J+1 & J & 1 \\ M & -M & 0 \end{pmatrix} \qquad (-1)^{J-M-1} \left[\frac{(J + M + 1)(J - M + 1) \cdot 2}{(2J + 3)(2J + 2)(2J + 1)} \right]^{\frac{1}{2}}$$

$$\begin{pmatrix} J & J & 1 \\ M & -M-1 & 1 \end{pmatrix} \qquad (-1)^{J-M} \left[\frac{(J - M)(J + M + 1) \cdot 2}{(2J + 2)(2J + 1)(2J)} \right]^{\frac{1}{2}}$$

$$(J, J, 1)$$

$$\begin{pmatrix} J & J & 1 \\ M & -M & 0 \end{pmatrix} \qquad (-1)^{J-M} \frac{M}{[(2J + 1)(J + 1)J]^{\frac{1}{2}}}$$

② Zeeman effect

$$(\gamma' \; j_1' \; j_2 \; J' || \mathbf{T}(k) || \gamma \; j_1 \; j_2 \; J) \tag{7.1.7}$$

$$= (-1)^{j_1' + j_2 + J + k} [(2J + 1)(2J' + 1)]^{\frac{1}{2}} \begin{Bmatrix} j_1' & J' & j_2 \\ J & j_1 & k \end{Bmatrix} \times (\gamma' \; j_1' || \mathbf{T}(k) || \gamma \; j_1)$$

$$\begin{Bmatrix} a & b & c \\ 1 & c-1 & b-1 \end{Bmatrix} = (-1)^s \left[\frac{s(s+1)(s-2a-1)(s-2a)}{(2b-1)2b(2b+1)(2c-1)2c(2c+1)} \right]^{\frac{1}{2}}$$

$$\begin{Bmatrix} a & b & c \\ 1 & c-1 & b \end{Bmatrix} = (-1)^s \left[\frac{2(s+1)(s-2a)(s-2b)(s-2c+1)}{2b(2b+1)(2b+2)(2c-1)2c(2c+1)} \right]^{\frac{1}{2}}$$

$$\begin{Bmatrix} a & b & c \\ 1 & c-1 & b+1 \end{Bmatrix} = (-1)^s \left[\frac{(s-2b-1)(s-2b)(s-2c+1)(s-2c+2)}{(2b+1)(2b+2)(2b+3)(2c-1)2c(2c+1)} \right]^{\frac{1}{2}}$$

$$\begin{Bmatrix} a & b & c \\ 1 & c & b \end{Bmatrix} = (-1)^{s+1} \frac{2[b(b+1) + c(c+1) - a(a+1)]}{[2b(2b+1)(2b+2)2c(2c+1)(2c+2)]^{\frac{1}{2}}}$$

where $s = a + b + c$.

② Zeeman effect

$$(\gamma' \; j_1' \; j_2 \; J' || \mathbf{T}(k) || \gamma \; j_1 \; j_2 \; J)$$

(7.1.7)

$$= (-1)^{j_1' + j_2 + J + k} [(2J+1)(2J'+1)]^{\frac{1}{2}} \begin{Bmatrix} j_1' & J' & j_2 \\ J & j_1 & k \end{Bmatrix} \times (\gamma' \; j_1' || \mathbf{T}(k) || \gamma \; j_1)$$

$$(\gamma' \; j_1 \; j_2' \; J' || \mathbf{U}(k) || \gamma \; j_1 \; j_2 \; J)$$

(7.1.8)

$$= (-1)^{j_1 + j_2 + J' + k} [(2J+1)(2J'+1)]^{\frac{1}{2}} \begin{Bmatrix} j_2' & J' & j_1 \\ J & j_2 & k \end{Bmatrix} \times (\gamma' \; j_2' || \mathbf{U}(k) || \gamma \; j_2)$$

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