Coherence time of a quantum gas

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Plan

1. **Time coherence of a BEC**
   - Motivation
   - Considered System and observables
   - Previous results

2. **Bosons**
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   - Analytical results
   - Analytics and interpretation
   - How to measure and numbers

3. **Fermions**
   - Interest and difficulties
   - General theory
   - Example: the unitary Fermi gas
How long is the intrinsic coherence time of BEC?

**Fundamental Interest:**

Coherence is a key property of Bose-Einstein condensates

- **Spatial coherence** extends over the entire system size.
- **What about time coherence?**
  
  Trapped cold gas = isolated system
  
  Intrinsic coherence time of an isolated “macroscopic” quantum system

**Interest for Applications:**

Interferometric measurements (internal or external states)

Clocks with trapped atoms, e.g. on an Atom chip
Considered System

We consider: 3D system, finite size, $N$ fixed, $T < T_c$, condensate made of bosons or pairs of opposite spin fermions

Time coherence is described by the time correlation function

$$g_1(t, 0) = \langle \hat{\psi}^\dagger(r, t) \hat{\psi}(r, 0) \rangle \xrightarrow{t \to \infty} \mid \phi_0(r) \mid^2 \langle \hat{a}_0^\dagger(t) \hat{a}_0(0) \rangle$$

- $T = 0$: The correlation function oscillates (no decay) at a frequency $\mu_0/h$ Beliaev ‘60s
- $T \neq 0$: The correlation function decays
  thermal excitations = dephasing environment
**Observables of interest**

**Time correlation function of condensate amplitude** $\langle a_0^+(t)a_0(0) \rangle$:

Accessible by an interferometric measurement.

\[ \langle a_0^+(t)a_0(0) \rangle \]

**Modulus-phase representation**:

\[ a_0 = e^{i\theta} \sqrt{N_0}, \quad [N_0, \theta] = i \]

For a Gaussian probability distribution of $\theta(t) - \theta(0)$ and weak fluctuations of $N_0$

\[ |\langle a_0^+(t)a_0(0) \rangle| \sim \langle N_0 \rangle \exp \left\{ -\frac{1}{2} \text{Var} [\theta(t) - \theta(0)] \right\} \]

The correlation function decays as the variance of the accumulated phase grows $\text{Var} [\theta(t) - \theta(0)]$ as a function of time $t$. 
Previous predictions on phase spreading at $T \neq 0$

**Bosons**

  
  The phase spreading is **DIFFUSIVE**: $\text{Var} [\theta(t) - \theta(0)] \sim Dt$

- Many-body approach (but interactions among Bogoliubov modes were neglected) A. Kuklov, J. Birman '98

  The phase spreading is **BALLISTIC**: $\text{Var} [\theta(t) - \theta(0)] \sim \alpha t^2$

**Paired Fermions**: no predictions.
Bosons : Classical Field simulations

$\psi$ describes condensate and non-condensed fraction

\[ i\hbar \partial_t \psi = \left[ -\frac{\hbar^2 \Delta}{2m} + g |\psi(\vec{r}, t)|^2 \right] \psi \]

1. Stochastic fields $\psi_i(0)$ sampling $\sigma = \frac{1}{Z} e^{-\beta E[\psi, \psi^*]}$ (canonical ensemble)

2. Deterministic evolution with NLSE

3. Calculate the averages at time $t$. 
Amplitude correlation function

\[ N = 4 \times 10^5, \ T = 0.17T_c, \ \mu = 700, \ T_C = 18000 \ (\hbar^2/mV^{2/3}). \]

1) **The correlation function decays** \( \langle a^*_0(t)a_0(0) \rangle \rightarrow 0 \) **for** \( t \rightarrow \infty \)

**Trivial factor** \( \exp(i\rho gt/\hbar) \) **eliminated**
2) Phase difference $\theta(t) - \theta(0)$ spreads ballistically over a wide range of temperatures.
Correlation function of $\dot{\theta}(t)$ and phase variance

**Correlation function:**
$$C(t) = \langle \dot{\theta}(t)\dot{\theta}(0) \rangle - \langle \dot{\theta} \rangle^2$$

**Phase variance:**
$$\text{Var} [\theta(t) - \theta(0)] = 2t \int_0^t C(\tau) d\tau - 2 \int_0^t \tau C(\tau) d\tau$$

The longtime behavior of the phase depends on the limit $A$ of $C(t)$

- $\Rightarrow A = 0 \Rightarrow$ diffusive
- $\Rightarrow A \neq 0 \Rightarrow$ ballistic
Illustration by a classical field simulation

Correlation of the condensate phase derivative

\[ C(\tau) = \langle \dot{\theta}(\tau)\dot{\theta}(0) \rangle - \langle \dot{\theta} \rangle^2 \]

Neglecting interactions among quasi particles (Kuklov, Birman)

Ergodic theory

Quantum-optics type theory (Zoller, Gardiner, Graham)
Role of the Bogoliubov occupation numbers $n_k$

**Condensate phase derivative**
\[ \dot{\theta}(t) = \sum_k A_k n_k \]

**Bogoliubov:**
\[ \langle n_k(t)n_{k'}(0) \rangle - \langle n_k \rangle \langle n_{k'} \rangle = \delta_{k,k'} \langle n_k \rangle^2 \]

**Gaussian model or ME (Landau and Beliaev damping):**
The $b_k$ are Gaussian variables with a finite correlation time
\[ \langle n_k(t)n_{k'}(0) \rangle - \langle n_k \rangle \langle n_{k'} \rangle = \delta_{k,k'} \langle n_k \rangle^2 e^{-\Gamma_k t} \]

**Ergodic theory:**
The $b_k(t \to \infty)$ uniformly explore a fixed-energy surface
\[ \langle n_k(t \to \infty)n_{k'}(0) \rangle - \langle n_k \rangle \langle n_{k'} \rangle \neq 0 \]
Predictions of the ergodic model

- For a system prepared in the **canonical ensemble**, the correlation function of the phase derivative does not vanish at long times. (Different from quantum-optics like models).

\[
C(t) = \langle \dot{\theta}(t) \dot{\theta}(0) \rangle - \langle \dot{\theta} \rangle^2 \to A \neq 0 \text{ for } t \to \infty
\]

- The condensate **phase spreads ballistically**

\[
\text{Var} [\theta(t) - \theta(0)] \sim At^2
\]

- The coefficient \(A\) is given by the ergodic theory:

\[
A = \frac{1}{\hbar^2} \left( \frac{d \langle \mu \rangle_m c}{dE} \right)^2_{E=\bar{E}} \text{Var} H
\]

- **Ergodicity is ensured by the interactions among Bogoliubov quasiparticles.**

- **The ergodic model can be generalized to the quantum case**
Ergodic theory (1)

The long time limit of the correlation function $C(t) = \langle \dot{\theta}(t) \dot{\theta}(0) \rangle - \langle \dot{\theta} \rangle^2$ can be found with an ergodic theory

- **Replace the long time limit with the time average**

$$C_A(+) \equiv \lim_{t \to +\infty} \frac{1}{t} \int_0^t d\tau \left[ \langle A(\tau) A(0) \rangle - \langle A \rangle^2 \right].$$

- **Use the exact eigenstates**:

$$\rho = \sum_m p_m |m\rangle \langle m|$$

In the absence of degeneracies you get:

$$C_A(+\infty) = \sum_m p_m \langle m|A|m\rangle^2 - \left[ \sum_m p_m \langle m|A|m\rangle \right]^2.$$

- **“Eigenstate thermalization hypothesis”** The average of an observable $A$ on an eigenstate $|m\rangle$ is close to the microcanonical average

$$\langle m|A|m\rangle \simeq \langle A \rangle_{mc}(E_m)$$

Ergodic theory (2)

Applied to our case: correlation function of the phase derivative
\[ A = -\hbar \dot{\theta} \equiv \mu \]

\[ C_\theta(+\infty) = \sum_E p(E) \left[ \langle \mu \rangle_{mc}(E) - \langle \mu \rangle_{mc}(\bar{E}) \right]^2 = \left( \frac{d\langle \mu \rangle_{mc}}{dE} \right)^2_{E=\bar{E}} \]

Consequence on phase spreading:
\[ \text{Var} [\theta(t) - \theta(0)] \sim \frac{t^2}{\hbar^2} \left( \frac{d\langle \mu \rangle_{mc}}{dE} \right)^2_{E=\bar{E}} \]

Physical interpretation:
For a given realization of energy \( E \), at long times
\[ \theta(t) - \theta(0) \sim -\frac{\mu(E)t}{\hbar} \quad \Delta [\theta(t) - \theta(0)] \sim \frac{t}{\hbar} \left. \frac{d\mu}{dE} \right|_{E=\bar{E}} \Delta E \]
What happens in the micro canonical ensemble when $C_\dot{\theta}(+\infty) = 0$?

$$\text{Var}[\theta(t) - \theta(0)] = Dt \quad \text{phase diffusion} \quad \hbar D V/g = f\left(\frac{k_B T}{\rho g}\right)$$

Classical Kinetic equations

Quantitative test: classical kinetic theory on a lattice (with a cut-off)
Summary of results (weakly interacting bosons)

- Using ergodic theory we predict the long time behavior of the phase for a finite temperature condensate:

  \[ \text{Var}[\theta(t) - \theta(0)] \approx \mathcal{A} t^2 \quad \text{with} \quad \mathcal{A} \propto \text{Var} E \]

- To describe the approach to the long time limit, we write kinetic equations and explicitly calculate

  \[ \text{Var}[\theta(t) - \theta(0)] \approx \mathcal{A} t^2 + \mathcal{B} t + \mathcal{C} \quad \text{for} \quad t \to \infty \]

The coefficient \( \mathcal{B} \) (phase diffusion) sets the limit of phase coherence in a finite temperature BEC.


- Collaboration with P. Treutlein for the first experimental determination of the BEC coherence time.
How to measure the time correlation function

Initially all atoms are in state $a$
Ramsey scheme with very small angle pulse $\eta \ll 1$.
The first pulse is at time $0$, the second pulse is at time $t$.

\[
\hat{\psi}_a(r, 0^+) = \sqrt{1 - \eta^2} \hat{\psi}_a(r, 0^-) + \eta \hat{\psi}_b(r, 0^-) \\
\hat{\psi}_b(r, 0^+) = \sqrt{1 - \eta^2} \hat{\psi}_b(r, 0^-) - \eta \hat{\psi}_a(r, 0^-)
\]

- No interactions $a$-$b$ (reduce the spatial overlap)
- No interactions $b$-$b$ (very dilute cloud)

\[
\hat{b}_0(t^-) = e^{i\delta t} \hat{b}_0(0^+) \quad \delta = \omega_L - \omega_{ab} \quad \text{rotating frame at frequency } \omega_L
\]

After the second pulse measure: \[\langle \hat{b}_0^+(t^+) \hat{b}_0(t^-) \rangle\]
How to measure the time correlation function

For $\eta \ll 1$:

\[ N_{b0}(t) = 2\eta^2 \left\{ \langle \hat{n}_0 \rangle + \text{Re} \left[ e^{i\delta t} \langle \hat{a}_0^\dagger(t)\hat{a}_0(0) \rangle \right] \right\} + O(\eta^4) \]

- The signal is small: one must detect a small number of atoms
- The contrast is independent of $\eta$ : equal to 1 at $t = 0$

By varying $\delta t \equiv (\omega_L - \omega_{ab})t$

\[ \frac{N_{b0}^{\text{max}} - N_{b0}^{\text{min}}}{N_{b0}^{\text{max}} + N_{b0}^{\text{min}}} = \frac{|\langle \hat{a}_0^\dagger(t)\hat{a}_0(0) \rangle|}{\langle \hat{a}_0^\dagger(t)\hat{a}_0(t) \rangle} = \text{contrast} \]

Example of Numbers for the ballistic phase spreading

$N = 4 \times 10^5$, $k_B T/\rho g = 3$, $a = 5.3\text{nm}$, $T = 0.3 T_c = 0.14 \mu\text{K}$ \Rightarrow $t_{br} = 750\text{ms}$

Blurring times of tens of ms are predicted in small samples.
Phase dynamics of a condensate of paired fermions

Interest of fermionic gases
Possibility to explore different interaction regimes $-\infty < \frac{1}{na^3} < +\infty$

→ Introduce the condensate phase operator and study its dynamics at $T \neq 0$
- No Bogoliubov theory available
- Two excitation branches F,B
- Interactions between excitations

General theory for phase coherence at $T \neq 0$ (fermions and bosons)
Phase operator and its time derivative

The condensate wave function $\phi(r_1, r_2)$ is the eigenvector of the 2-body density matrix $\rho_2(r_1, r_2; r'_1, r'_2) = \langle \hat{\psi}^\dagger(r'_1) \hat{\psi}^\dagger(r'_2) \hat{\psi}(r_2) \hat{\psi}(r_1) \rangle$ with macroscopic eigenvalue $\bar{N}_0$

The Condensate phase $\hat{\theta}_0$ is the phase of the operator $\hat{a}_0$ obtained by projecting the pairing field $\hat{\psi}(r_2) \hat{\psi}(r_1)$ onto $\phi(r_1, r_2)$.

The decay of the pair correlation function is then given by

$$g_1(t) = \langle \hat{a}^\dagger_0(t) \hat{a}_0(0) \rangle \sim \bar{N}_0 \langle e^{-i\hat{\theta}_0(t)} e^{i\hat{\theta}_0(0)} \rangle$$

Coarse grained average of phase derivative

$$-\frac{\hbar}{2} \frac{d\hat{\theta}_0}{dt} = \mu_0(\hat{N}) + \sum_{s=F, B} \sum_{\alpha} \frac{d\epsilon_{s,\alpha}}{dN} \hat{n}_{s,\alpha}$$

First microscopic derivation at the operator/multimode level
Using a generalized coherent state Ansatz including moving pairs (Blaizot-Ripka)
Decay of the time correlation function

The system state: \( \hat{\rho} = \sum_\lambda \Pi_\lambda |\psi_\lambda\rangle \langle \psi_\lambda | \) with \( \hat{H}|\psi_\lambda\rangle = E_\lambda |\psi_\lambda\rangle \)

Correlation function in a many-body eigenstate \( \psi_\lambda \)

\[
g_1^\lambda(t) \approx \bar{N}_0 e^{iE_\lambda t/\hbar} \langle \psi_\lambda | e^{-i(\hat{H} + \hat{W})t/\hbar} |\psi_\lambda\rangle \\
\hat{W} = \hbar \frac{d\hat{\theta}_0}{dt} + O\left( \frac{1}{N} \right)
\]

Leading term for the decay of \( g_1^\lambda(t) \)

\[
\langle \psi_\lambda | \frac{d\hat{\theta}_0}{dt} |\psi_\lambda\rangle \xrightarrow{ETH} \mu_{mc}(N_\lambda, E_\lambda)
\]

Gaussian decay of \( g_1 \) after ensemble average (ballistic phase spreading) \( t_{br} \propto N^{1/2} \).

Subleading term quadratic in \( \hat{W} \)

\[
\gamma_\lambda + i\delta_\lambda = \int_0^{+\infty} dt \left\{ \left\langle \frac{d\hat{\theta}_0(t)}{dt} \frac{d\hat{\theta}_0(0)}{dt} \right\rangle_\lambda - \left\langle \frac{d\hat{\theta}_0}{dt} \right\rangle_\lambda^2 \right\}
\]

Exponential decay of \( g_1 \) even in the microcanonical ensemble (phase diffusion) \( t_{\text{diff}} \propto N \).
Proposed experimental configuration and protocol

- Ramsey interferometry to measure $g_1(t)$: two pulses of small angle separated by time $t$
- **Dimerize the pairs for preparation, pluses and detection**
- $\langle n_{\text{sec}} \rangle$ oscillates at $\omega = 2(\mu_{\text{main}} - \mu_{\text{sec}})/\hbar$, the contrast is $|g_1(t)/g_1(0)|$
Thermal blurring time of a unitary Fermi gas ($a = \infty$)

From equation of state measurements (Salomon LKB, Zwierlein MIT)

Canonical ensemble $T = 0.12 T_F \simeq 0.7 T_c$, $N = 10^5$, $T_F = 1\mu K$

Then: $t_{br} = 20\text{ms}$
In a finite size condensed system at non-zero temperature, the time correlation function decays even when the system is isolated in its evolution and the particle number $N$ is fixed.

The loss of phase memory is due to interactions of the condensate with the excited modes that constitute a dephasing environment.

$\hat{\theta}_0$ evolves at long times as $-2\mu_{mc}(E)t/\hbar$ where $\mu_{mc}(E)$ is the microcanonical chemical potential; energy fluctuations from one realization to the other then lead to a ballistic spreading of the phase and to a Gaussian decay of the temporal coherence function with a characteristic time $\propto N^{1/2}$.

In the absence of energy fluctuations, the coherence time scales as $N$ due to the diffusive motion of $\hat{\theta}_0$.

We predict a coherence time of tens of milliseconds for the canonical ensemble unitary Fermi gas.